



## Cantor's paradise from the perspective of non-revisionist Wittgensteinianism

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### ABSTRACT

**Cantor's paradise from the perspective of non-revisionist Wittgensteinianism:** Ludwig Wittgenstein is known for his criticism of transfinite set theory. He forwards the claim that we tend to conceptualise infinity as an object due to the systematic confusion of extension with intension. There can be no mathematical symbol that directly refers to infinity: a rule is the only form by which the latter can appear in our symbolic operations. In consequence, Wittgenstein rejects such ideas as infinite cardinals, the Cantorian understanding of non-denumerability, and the view of real numbers as a continuous sequence of points on a number line. Moreover, as he understands mathematics to be an anthropological phenomenon, he rejects set theory due to its lack of application. As I argue here, it is possible to defend Georg Cantor's theory by taking a standpoint I call quietistic conventionalism. The standpoint broadly resembles Wittgenstein's formalist middle period and allows us to view transfinite set theory as a result of a series of definitions established by arbitrary decisions that have no ontological consequences. I point to the fact that we are inclined to accept such definitions because of certain psychological mechanisms such as the hypothetical Basic Metaphor of Infinity proposed by George Lakoff and Rafael E. Núñez. Regarding Wittgenstein's criterion of applicability, I argue that it presupposes a static view of science. Therefore, we should not rely on it because we are unable to foresee what will turn out to be useful in the future.

### KEYWORDS

transfinitism; set theory; conventionalism; Georg Cantor; Ludwig Wittgenstein

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Ludwig Wittgenstein's reply to the well-known remark by David Hilbert about Georg Cantor's transfinite paradise suggests that mathematicians fall victim to an illusion: they should rather perceive the alleged paradise as hell in disguise. Wittgenstein comments:

I would say, "I wouldn't dream of trying to drive anyone from this paradise." I would do something quite different: I would try to show you that it is not a paradise — so that you'll leave of your own accord. I would say, "You're welcome to this: just look about you" (LFM: 103).

In this paper, I postulate that Cantor's set theory is neither a paradise nor a hell of mathematics, but rather "plain" ground, just as imaginary numbers or non-Euclidean geometry. A Wittgensteinian (non-revisionist) critique should aim to make the fact that one can perform symbolic operations on transfinites and perhaps even utilise them as a novel technique for approaching certain mathematical problems, avoiding entanglement in confusing metaphysical standpoints, and versions of Platonist realism in particular, explicit.

Such an approach to the problem calls for a correction of the prevalent but misleading realist interpretation of set theory while simultaneously rejecting many of Wittgenstein's revisionist remarks on Cantor, real numbers, and transfinites. Wittgenstein does indeed present a certain intriguing conception of mathematics, though he is apparently unaware of breaching the border between mere clarification and mathematical theorising, assuming we can speak of such a strict demarcation.

It is noticeable that a similar neutral stance was adopted for practical reasons by leading scholars of the Lvov-Warsaw School of philosophy and the Warsaw School of mathematics in the 1920s and 1930s (Woleński, 1985: 179–185). They were willing to examine different axiomatic systems regardless of their own attitudes towards various philosophical insights those systems carried, since they were understood to be personal opinions. This disposition is illustrated by the following quotation on the axiom of choice from Waław Sierpiński's *Cardinal and ordinal numbers*:

Still, apart from our being personally inclined to accept the axiom of choice, we must take into consideration, in any case, its role in the Set Theory and in the Calculus. On the other hand, since the axiom of choice has been questioned by some mathematicians, it is important to know which theorems are proved with its aid and to realize the exact point at which the proof has been based on the axiom of choice [...]. And after all, even if no one questioned the axiom of choice, it would not be without interest to investigate which proofs are based on it and which theorems can be proved without its aid — this, as we know, is also done with regard to other axioms (Sierpiński, 1958: 90).

This paper is divided into four sections, the first of which gives an outline of transfinite mathematics and its history. It is followed by an attempt at a reconstruction of Wittgenstein's criticism of the notion of transfinite mathematics in §2. Next, in §3 I lay down a non-realist reading of transfinite mathematics, which I consider to be non-revisionist Wittgensteinism. To this end I utilise, though not uncritically, George Lakoff and Rafael E. Núñez's idea of the Basic Metaphor of Infinity. Finally, in §4 I analyse Wittgenstein's stance, focusing mainly on the applicability of various mathematical techniques.

## §1

The problem of infinity in mathematics is certainly much older than Cantor's set theory. It is impossible to determine the specific moment when *Homo sapiens* recognised that the series of natural numbers continues indefinitely. The ancient Greeks were no doubt aware of this, as they possessed the concept of infinity (in §3 I will present a possible causal explanation for its emergence). When Aristotle famously differentiated between two general ways of understanding this concept, he was motivated to do so by concern for the problems it might create, an approach which he shared with many of his predecessors, including Plato, the Eleatic school, and Pythagoras. In search for the sole intelligible construal of infinity, the Philosopher decided to reject the understanding of infinity as "actual", that is, somehow achievable and graspable magnitude, in favour of what is called "potential infinity", that is, the notion of a perpetual process, extending towards the horizon of our comprehension.

Despite Aristotle's objections, the concept of actual infinity was successfully employed by Archimedes in his famous proof that the ratio of the area of a given sphere to the total area of a cylinder enclosing it is precisely  $2/3$ . It could be argued that if Archimedes were to share Stagirite's doubts, he would be unable to determine that the ratio is a rational number, since in the course of an infinite process that number would never be reached.

In early modernity, a new type of mathematical infinity was introduced: the infinitesimals postulated by differential and integral calculus. The definition of infinitesimals given by Leibniz clearly referred to the understanding of infinity as actual. By that time the concept had been rehabilitated, initially by theologians — who had construed it as the fullness of being and an aspect of divinity — and subsequently in epistemology and mathematics by Nicholas of Cusa, René Descartes, Blaise Pascal, and others. On the other hand, the British Empiricists Berkeley and Hume took a strictly finitist approach.

However, infinity in calculus could also be understood as a convenient representation of an infinite process, therefore as a potential infinity. This became evident in the nineteenth century when Augustin-Louis Cauchy and

Karl Weierstrass provided more rigorous foundations for analysis. Defining the limit of a sequence and uniform convergence allowed them to rid calculus of troublesome actual infinities.

In the latter half of the nineteenth century, another breakthrough in mathematics occurred, this time supporting the concept of actual infinity. The study of real numbers and the functions defined on them led Weierstrass to propose his peculiar continuous non-differentiable function while Richard Dedekind produced a method of constructing non-rational real numbers. Yet most crucially Cantor, departing from Bolzano's definition of a set, developed a theory allowing for the description of the infinite series of different orders of infinity.

Cantor's article from 1874 — the first instance of the formulation of the notion of transfinite numbers — introduced the concept of denumerable infinity by means of a bijective function placing the elements of a given set in a one-to-one correspondence with the elements of the set of all natural numbers  $\mathbf{N}$ . One-to-one correspondence was utilised by him as a decisive argument in favour of the denumerability of the set of real algebraic numbers; that is, all roots of non-zero polynomials in one variable with rational coefficients. Subsequently, Cantor presented a proof that any given interval of real numbers  $[a, b]$  contains infinitely many transcendental numbers; that is, non-algebraic real numbers.<sup>1</sup> On this he based the claim that no set of real numbers belonging to an interval of real numbers can have one-to-one correspondence with  $\mathbf{N}$ , and thus the set of real numbers  $\mathbf{R}$ , the so-called continuum, is non-denumerable, unlike the set of all real algebraic numbers. He concluded that the continuum must be a magnitude greater than the infinity describing the set of all natural numbers.

In his 1878 paper, Cantor introduced the concept of a set's cardinality: any two sets are of the same cardinality when there is a one-to-one correspondence between them. He also noted that any countably infinite product of copies of  $\mathbf{R}$  has the same cardinality as  $\mathbf{R}$  itself. In other words, there is a one-to-one correspondence between the points on any given line segment and the points in an  $n$ -dimensional space for any positive integer  $n$ . This article was also where the famous Continuum Hypothesis (CH) was put forward: Cantor conjectured that there is no set whose cardinality is exactly between that of integers and real numbers.<sup>2</sup>

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<sup>1</sup> The proof postulates the construction of a hierarchy of ever smaller nested intervals to demonstrate that as this hierarchy approaches infinity it produces a transcendental number. There is no agreement among mathematicians whether this proof is really constructive. Among those opting for its constructivity were Abraham Fraenkel (Fraenkel, 1930) and Barnaby Sheppard (Sheppard, 2014); meanwhile, those who opposed were Ian Stewart (Stewart, 2015) and Lech Gruszecki (Gruszecki, 2005: 168), among others.

<sup>2</sup> In this outline (save for this footnote), I consciously neglect to treat upon the problem of the well-ordering of transfinite sets. The CH requires the assumption that all transfinite

Cantor's most important contributions were contained in his 1891 paper, containing both a formulation and a proof of Cantor's theorem and his famous diagonal argument. The former states that the power set of any given set  $A$  (the set of all subsets of  $A$ ) necessarily has a greater cardinality than  $A$ . Most importantly, this also holds true for infinite sets. For example, assuming that the cardinality of  $\mathbf{N}$  is  $\aleph_0$ , the cardinality of its power set is equal to  $2^{\aleph_0}$ , but is also strictly greater than  $\aleph_0$ . This means that the power set of  $\mathbf{N}$  is uncountable. The cardinality of the continuum ( $\mathbf{R}$ ) is the cardinality of  $2^{\aleph_0}$ , which is written with the symbol  $c$ . Also, since the power of  $\mathbf{R}$  is equal to the power of  $\mathbf{R}^n$  for any integer  $n$ , then  $c = c^n$  for any integer  $n$ . Since, according to the CH,  $c$  is the next cardinal number after  $\aleph_0$ ,  $c = \aleph_1$ .  $c$  is, of course, not the greatest cardinal number, for it follows from Cantor's theorem that the power set of  $\mathbf{R}$  has a cardinality strictly greater than  $c$  (namely,  $\aleph_2$ , the third transfinite cardinal; again, assuming the truth of the CH). The power set of that power set must have an even greater cardinality, and so on. This introduces an infinite hierarchy of transfinite cardinals corresponding to the infinite hierarchy of power sets.

Similar to the argument for the existence of infinitely many transcendental numbers, the diagonal argument takes upon itself to prove that the set of real numbers cannot be put into a one-to-one correspondence with the set of natural numbers, and that in consequence the cardinality of the former is strictly greater than the cardinality of the latter. The original formulation of the argument from Cantor's 1891 paper describes the set of all infinite binary sequences, referred to as  $T$ . The set  $T$  cannot be denumerated, for there will always be a sequence within it not included in any denumeration: we can always construct such a sequence by selecting the  $n$ -th digit from each  $n$ -th sequence in  $T$  and inverting it (changing 0 to 1 and vice versa). The resulting sequence differs from any sequence belonging to  $T$  by at least one digit. (Were one to write the sequences as rows of a table, each column containing specific digits from all sequences, the digits to be inverted would form a diagonal line across the table; hence, the name of the argument.) This allowed Cantor to easily prove the uncountability of the set  $T$  by contradiction. Furthermore, since we may assume that all real numbers can be represented with infinite binary sequences divided in some place with a radix point, there is a one-to-one correspondence between  $T$  and  $\mathbf{R}$ , meaning that  $\mathbf{R}$  is uncountable.

In the early twentieth century, the original rendition of set theory — later named naïve set theory — had been proven to yield antinomies. Several attempts had been made to remedy this, one of them being the axiomatisation proposed by Ernst Zermelo and Abraham Fraenkel, today considered to be the standard version of the theory. Strictly speaking, there are two forms

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sets are well-ordered. Cantor attempted to prove this by introducing infinite ordinals; he was, however, only able to prove that countably transfinite sets were well-ordered.

of axiomatic set theory: the narrower (ZF) containing eight axioms, and the wider (ZFC) including also the axiom of choice (AC). This additional axiom attracted some controversy as it assumes the existence of non-construable objects and yields counter-intuitive results, including the Banach-Tarski paradox.<sup>3</sup> Apart from AC, another intriguing element of ZF and ZFC is the axiom of infinity, warranting the existence of at least one infinite set of cardinality equal to the cardinality of  $\mathbb{N}$ .

The reception of Cantor's achievements by his contemporaries was mixed, and it remains so today. During his lifetime, he had a powerful opponent in Leopold Kronecker, a proponent of Aristotelian finitism. On the other hand, transfinite methods were acclaimed by distinguished mathematicians of the time: Weierstrass and Dedekind. Mathematics at the beginning of the twentieth century was dominated by the debate regarding the validity of non-constructive proof in the transfinite domain. Today, most mathematicians accept Cantor's accomplishments, along with some version of the Platonist ontology of mathematics, although constructivism and finitism in their many denominations are not without adherents. The anecdotal evidence that I have gathered suggests that transfinite methods are more likely to be rejected by scholars of applied mathematics who tend to disregard it as fantasy or theology, a pejorative term in this context.

One should also note that CH is not the only element of Cantor's set theory, which must be settled by means of a mathematician's decision. The relation of equinumerosity, established by means of one-to-one correspondence between various denumerable infinite sets, does not hold within the Alpha-Theory, an alternative approach to non-standard analysis, which upholds Euclid's principle that the whole is larger than its parts, also in the domain of infinities. By means of Alpha-Theory we can demonstrate that there is no equinumerosity between the set of natural numbers  $\mathbb{N}$  and its infinite proper subsets like the set of even numbers or the set of primary numbers: their cardinality is lesser than the cardinality of  $\mathbb{N}$ ; likewise, we can show that the cardinality of the set of rational numbers is greater than the cardinality of  $\mathbb{N}$  (Benci & Di Nasso, 2014). Thus, we may choose between very disparate transfinite calculi.

## §2

Although the evolution of Wittgenstein's thought was a continuous and complicated process, having different dynamics of development in its various planes,

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<sup>3</sup> The so-called paradox is a theorem provable in ZFC, according to which it is possible to disassemble a given 3-dimensional sphere into a finite number of disjoint subsets and reassemble them into two spheres, identical to the original in size (Banach & Tarski, 1924). The majority of contemporary mathematicians no longer consider it to be a paradox (Stromberg, 1979).

and although the popular belief that there had been the “early” and “late” Wittgenstein is rather an artefact of his interpreters than a reflection of reality,<sup>4</sup> in some particular fields such a periodisation may prove useful, with certain reservations considered. Regarding his philosophy of mathematics, three (and a half) distinctive stages of development are discernible. The first, the Tractarian stage, shall not be dealt with here. Wittgenstein's post-Tractarian philosophy can be divided into two general periods, and so can his consideration of mathematical problems.

During the first period, he was an advocate of understanding mathematics as a system of very strictly understood calculi, the identity of which boil down to the set of their rules. (At the onset of this stage, he developed a peculiar verificationist approach to mathematics; this stands for the abovementioned “half”). During the second period, Wittgenstein focused his attention on the form of life and on practice, that is: the role operations on mathematical symbols play in extra-mathematical activities and language games. While interpreters vary in their identification of the moment of Wittgenstein's transition from the calculus period to the language-game period, it can be assumed that this occurred around 1935. In any case, the transformation was by no means radical. One may point to the common trait of the two periods: the rejection of the objectivist vision of mathematical reality held by Bertrand Russell and Godfrey H. Hardy, according to which mathematics was to transcend the symbolical capabilities of human beings (Floyd, 2005: 112; Gerrard, 1991: 127–131).

#### THE CALCULUS CONCEPTION

One may say, following Steve Gerrard, that the calculus period is founded upon a radical understanding of the autonomy of syntax principle, according to which the evaluation or justification of rules is pointless, since it is not possible to compare different calculi or even to say anything about them in general. Even the slightest change in the set of the rules results in the establishment of another calculus; syntax is self-sufficient, as it does not need to be rooted in any reality, including that of human activity, and any attempt to provide a grounding for it or to prove its consistency, like that undertaken by advocates of set theory, makes no sense (WVC: 119–121). Within such a framework, consistency itself becomes quite a trivial property: in the early 1930s, Wittgenstein had an extremely liberal approach to the problem of contradiction. He even suggested that too much effort is made to erase it from calculus. (Such a lax attitude may be regarded as a flaw.) Among the undoubtable weaknesses of Wittgenstein's thought during the calculus period were the inability

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<sup>4</sup> I deal with this issue in my paper *The other Wittgenstein* (Gomułka, 2019).

to formulate any theoretical account of the changes and development of calculi and a lack of clarity concerning the problem of rules<sup>5</sup> (Gerrard, 1991: 133–136; Gomułka, 2019).

The information that we have regarding the first year of Wittgenstein's work after his return to philosophy in March 1928 is fragmentary and insufficient. However, the problem of infinity was undoubtedly among the first topics with which he engaged. Initially, as he collaborated with Frank Ramsey, he intended to supplement the conception of the *Tractatus* with the understanding of infinity that they both considered appropriate.<sup>6</sup> Together, they came to the conclusion that the requirements for a construction which is to be responsible for the popular notion of infinity are contradictory, for such a thing must be both complete and without an end.

In his *Philosophical remarks*, Wittgenstein forwards the claim that we tend to conceptualise the existence of actual infinity due to the systematic confusion of the two distinct meanings of the word “possibility”: the modal and the grammatical one. The former relates to the word “real”; hence in the modal sense, everything which is possible could be real. The latter meaning, deeply related within the concept of infinity, is not an attribute of objects or situations being described, but a feature of the operation of the syntax of calculus (PR: §141).

These two different meanings of the word “possibility” are related to the two different ways of proving formulas: regular and by induction. At the very beginning of his calculus period, Wittgenstein abandoned the belief expressed in the *Tractatus* that mathematical expressions (equations) are senseless pseudo-propositions (TLP: 6.2). He did so by means of the verification principle applied to formal sciences: if a proof is the method of establishing the truth-value of a mathematical theorem, then this proof should be understood to be the sense of that theorem, meaning that the theorem should be

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<sup>5</sup> The problem of rules was crucial for the calculus period. While Wittgenstein's conception still demanded absolutely strict and unambiguous rules, there were no grounds for meeting these requirements. After Wittgenstein discovered the limitations of logic, understood in the *Tractatus* as the transcendental base responsible for the functioning of the machinery of any meaningful symbolism, he at first tried to pursue a peculiar form of phenomenological theory which assumed the so-called *primäre Sprache*; that is, a level of language directly mirroring the structure of phenomena. However, that theory was short-lived due to its inner contradictions and was abandoned by him in October 1929 (Monk, 2014). Wittgenstein's later attempts would sometimes even approach Cartesian mentalism (WLC2: 36f.), which demonstrates his helplessness in the face of the problem.

<sup>6</sup> It is not clear how exactly Wittgenstein became aware of the necessity of such a correction. Ramsey's account from June 1929 suggests that he rejected his previous formulation as a result of considering general propositions (Moore, 1993: 48). Some authors suggest that Wittgenstein changed his stance in response to Brouwer's lecture (Wrigley, 1989; Holton & Price, 2003); a number of others believe that Wittgenstein's finitism was a result of Ramsey's influence (Sahlin, 1997; Marion, 1998: 84–109; Methven, 2015: 198–230).



recognised as a fully-developed proposition.<sup>7</sup> The Viennese philosopher went as far as to say that the symbol of a mathematical theorem is an abbreviation of its full proof; he did not, however, extend this to theorems proven by induction which he called “recursive”, since he did not consider them to be propositions, because, as he claimed, induction as such neither asserts nor negates anything (WVC: 82). A proof by induction should be taken as a certain template, a “signpost” indicating how the proof of a specific proposition is to be constructed (MS-105: 87).<sup>8</sup> Expressions within theorems proven by induction or even the recursive proofs themselves do not represent infinity, but rather simply point towards the infinite possibility enabled by grammar (WLC2: 14; MS-111: 46f.). In consequence, neither the fundamental theorem of algebra (MS-105: 101), nor Fermat’s Last Theorem (MS-107: 83)<sup>9</sup> can be considered valid mathematical propositions.

Wittgenstein’s grammatical finitism, according to which infinity is “sewn” into the syntax with no symbol being able to refer to it directly, entails a rejection of Cantor’s reading of the diagonal argument and the very idea of continuum, but also results in a peculiar conception of real numbers. Infinity is neither a number nor a magnitude (MS-106: 197), though that does not result from our cognitive limitations, but rather from the fact that the methods and criteria by which we determine finite magnitudes cannot be meaningfully employed to infinity (WVC: 187n). For Wittgenstein, infinity is never an object, but always given by induction; that is, by a rule.<sup>10</sup> He calls the idea of an infinite totality “utter nonsense”; since such an idea cannot be presented within any valid symbolism, Cantor and others had to postulate a fictitious one (PR: §§145, 174). Wittgenstein argues that no valid symbolical structure is able to demonstrate that the function  $f(m) = 2n$  from natural numbers to even numbers establishes a one-to-one correspondence between a set and its proper subset; it is just

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<sup>7</sup> While developing his views on calculus, Wittgenstein was inspired by Carl Johannes Thomae’s formalism and modified his views accordingly: since the process of proving deals with symbols within a certain calculus which is determined by a certain rule set, the meaning of those symbols is fully determined by the rules (WVC: 103–105).

<sup>8</sup> Marion points out that this idea was inspired by Thoralf Skolem’s primitive recursive arithmetic (Marion, 1998: 98).

<sup>9</sup> Wittgenstein’s comments on Fermat’s Last Theorem in his 1933 *Big typescript* suggest that he thought that theorem is somewhat external to the calculus (BT: 420).

<sup>10</sup> According to Marion, by claiming that the difference between the finite and the infinite is grammatical in nature, Wittgenstein takes a position which cannot be validly described as “finitist”. Marion prefers the term “intensionalism” as opposed to “extensionalism” to which all well-known standpoints (logicism, formalism, finitism, intuitionism, etc.) are subsumed (Marion, 1998: 186–189). It is also noticeable that Wittgenstein’s “intensionalism” is built upon the anti-logicist idea that we cannot talk of a class that has not been already effectively determined by a function, a series of forms, or simply the enumeration of its elements, present in the *Tractatus* (TLP: 5.501).

a sign for an infinite process (PR: §141), meaning that one is not in the position to talk about equinumerability in such cases.

According to Wittgenstein, we cannot determine whether real numbers are non-denumerable, since any real number *de facto* occurring in any of our calculi is determined by a certain law, and it is this law and not the number's extension (decimal or other) that determines the identity of a given real number. Examples of such laws include the ratio of a square's diagonal to its edge, the ratio of the area of a circle to the area of a square with an edge equal in length to the radius of that circle, etc. (PR: §§181, 186; WVC: 71, 81f.; PG: 484f.). To be able to determine that a number is non-rational, we have to lay down an adequate law. Thus, there is no repository with the magnitude of continuum. Dedekind's cut is simply an illusion: the supposed transcendental number can never be reached (MS-106: 78; PR: §181; BT: 496). It is precisely for this reason that Cantor's diagonal argument cannot prove the existence of non-countable real numbers: it is nothing but an algorithm returning a number different from all the numbers known to us up to this point. Since we may only provide a finite amount of numbers, and this limitation is not factual (epistemic) but grammatical in nature, by applying the diagonal method we simply obtain more numbers that we can add to our new, still finite, list.

By equating real numbers with laws, Wittgenstein burned the main bridge leading to actual infinite structures. Consequently, he was forced to reject certain received assumptions, such as the continuity of a function. This was, however, a price he was willing to pay, for above all he was opposed to what he considered to be a confusion of extensions, represented by lists, with intensions represented by laws.<sup>11</sup> Wittgenstein believed that such a perspective was deceitful "prose" added to calculus, allowing for alleged discoveries of non-rational points, which in reality may only be constructed (WVC: 129, 149, 175). All that is required is a calculus that works and needs only to be provided with rules and induction. Thanks to the latter, we know that we can execute certain operations indefinitely, but induction can only provide us with as much as it does and nothing more. When we divide  $1/3$  on paper, we can always obtain only a finite decimal expansion. "Everything else that is nonetheless *said*, e.g. that infinitely many threes follow, does not belong to mathematics proper but is a private matter" (WVC: 33; cf. BT: 466).<sup>12</sup>

<sup>11</sup> In *Philosophical grammar* we read: "After all I have already said, it may sound trivial if I now say that the mistake in the set-theoretical approach consists time and again in treating laws and enumerations (lists) as essentially the same kind of thing and arranging them in parallel series so that one fills in gaps left by another" (PG: 461).

<sup>12</sup> The account of the calculus phase of Wittgenstein's thought that I give may have been too simplified and fails to reflect the dynamics of this period. Some prominent interpreters argue against the verificationist reading of the middle period of his philosophy of mathematics (Floyd, 2000: 244–252).

## THE LANGUAGE-GAME CONCEPTION

While the transition from a paradigm focusing on calculus to one based on language-game was fluent in Wittgenstein's thought, several important features differentiating the latter from its predecessor can be identified. First, grammar, rules, and meaning become treated much less restrictively. Second, much more emphasis is placed on the practice of community activities, particularly on their feasibility. From the first two follows a third: the formalist autonomy of the syntax principle (in Carl Johannes Thomae's spirit) is significantly weakened. All three changes result from Wittgenstein's reflection on the workings of grammar and his engagement with the problem of the criteria of operation according to a given rule.

The less stringent treatment of grammar resulting from a greater emphasis on pragmatics, perhaps counterintuitively, led to a decline of Wittgenstein's *laissez-faire* attitude towards contradiction. As has been pointed out, by Gerrard (Gerrard, 1991: 136–139) among others, Wittgenstein recanted his earlier thesis that contradiction is not a problem for calculus, asserting that for us to recognise something as belonging to mathematics it must operate as such; that is, to be applicable to our everyday activities. Moreover, its application must be recognisable as a calculation.<sup>13</sup> In general, however, Wittgenstein became much more reserved with his resolute assertions regarding the nonsensicality of certain utterances. The paradigm which formed in the latter half of the 1930s beckoned him to search for the proper sense of expressions which several years before he would have simply considered nonsensical. A certain change of tone in his opinions on set theory is also noticeable. While expressions of sheer condemnation are easy to find in texts written during the calculus period,<sup>14</sup> compilations representing his later views, such as *Remarks on the foundations of mathematics* or *Lectures on the foundations of mathematics* (and their corresponding manuscripts) often express the spirit of PI §124, declaring that philosophy should not interfere with what Wittgenstein considers to be actual mathematics, but rather remain descriptive and leave it as it is (RFM: II, §62).

<sup>13</sup> See also Wittgenstein's well-known example of wood sellers (RFM: I, §149), discussed by David Cerbone (Cerbone, 2000).

<sup>14</sup> For example: "Set theory is wrong because it apparently presupposes a symbolism which doesn't exist instead of one that does exist (is alone possible). It builds on a fictitious symbolism, therefore on nonsense" (PR §174). Also: "A class cannot be finite or infinite. The words 'finite' and 'infinite' do not signify a supplementary determination regarding 'class'" (WVC: 102). The latter stance, expressed at the meeting of the Vienna Circle on June 19, 1930, was later replaced by the view that the concepts of infinite and finite classes do indeed have meanings, only their syntax differs substantially (WVC: 227f.; MS-113: 88r, repeated in: TS-211: 652; TS-213: 744r; PG: 464f.; and BT: 492). This appears to be one of the signs of Wittgenstein's shifting his initial verificationist stance towards the fully developed calculus view inspired by the Frege-Thomae dispute.

However, Wittgenstein did not withdraw any of his principal objections to transfinite set theory; on the contrary, they were reinforced by yet another crucial argument. First, as has been noted by Victor Rodych, Wittgenstein's stance regarding the intensional character of infinity had "not changed one iota" (Rodych, 2000: 289), and for that reason he still rejected the extensional reading of the concept. As a consequence, he retained the same negative attitude towards such ideas as infinite cardinals, the Cantorian understanding of non-denumerability, and the view of real numbers as a continuous sequence of points on a number line (RFM: II, §§32f.).

Second, Wittgenstein continued to deny the very need for a single foundational theory for the entirety of mathematics, which set theory was an attempt at doing.<sup>15</sup> He considered mathematics to be the sum of the various techniques developed during its history and did not recognise the need to subsume that colourful variety under this or that axiomatisation. Thus, while Quine and others tended to consider set theory as a useful foundation for mathematics itself (Rodych, 2000: 310), the Viennese philosopher rejected the very need to which they were responding (RFM: III, §§45f.).

Third, what Wittgenstein identifies as questionable in the works by Cantor and his followers is not the calculus itself, but the "prose" attached to it; that is, the allegedly illegitimate interpretation according to which calculus could be used to obtain a hierarchy of actual infinities. Likewise, Wittgenstein retained the belief that it is merely unjustified "prose" to claim that the diagonal argument is proof that the cardinality of the set of real numbers is strictly greater than the cardinality of the set of natural numbers; that we may speak of the continuum as a set and compare it with its proper subsets; and so on. Wittgenstein puts a lot of effort into distinguishing between what he considers to be prose and proper calculi, although he strips set theory of all its charm by doing so, for it was designed precisely to enable theorising about transfinite magnitudes (and this is also the reason why we remain captivated by it). Nonetheless, set theory presented without what Wittgenstein understands to be ambiguities amounts to a modest and uninteresting set of techniques.<sup>16</sup>

Another of Wittgenstein's objections to set theory, which originated only during his language-game phase, is based on the understanding of mathematics as an anthropological phenomenon, a human activity governed not only by the rules of grammar, but also directed by the practical goals at which it aims

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<sup>15</sup> The roots of that resistance can be traced back to the pre-Tractarian stage of his thought (Floyd, 2000: 233f.).

<sup>16</sup> Ryan Dawson (Dawson, 2016) took upon himself the task of reconstructing what would have become of set theory were it to look for Wittgenstein's approval during its development (not dealing with the problem of extra-mathematical applicability, which will be treated upon shortly within the main text of this paper).

(RFM: VII, §33). As Wittgenstein understands, the only reason for inventing, developing, and employing mathematical techniques is to rely on them over the course of our lives (LFM: 82).<sup>17</sup> According to this view, set theory is denied a place within mathematics, since it is nothing but a game of signs with no application. Even if we graciously permitted set theorists to use symbols allegedly referring to inconstruable extensions, their works would have no meaning outside of the system within which they operate:

*“These considerations may lead us to say that  $2^{\aleph_0} > \aleph_0$ .”*

That is to say: we can *make* the considerations lead us to that.

Or: we can say *this* and give *this* as our reason.

But if we do say it — what are we to do next? In what practice is this proposition anchored? It is for the time being a piece of mathematical architecture which hangs in the air, and looks as if it were, let us say, an architrave, but not supported by anything and supporting nothing (RFM: II, §35).<sup>18</sup>

For obvious reasons, set theory in its truncated form, stripped of the appeal of actual infinity, cannot have any external use, either. The only certain application that set theory may claim to have is its role as the foundation of mathematics, which Wittgenstein decisively rejected at all stages of his philosophical career. The theory is not only uninteresting, says the Viennese philosopher, but it is also disconnected from our other practices, mathematical or not. To use one of Wittgenstein's famous metaphors, it is like a cog detached from the working mechanism, spinning idly and aimlessly.

As will be shown in §4, the above argument against set theory has a certain weakness. Meanwhile, in §3 I shall leverage the differences between revisionist and non-revisionist traits in Wittgenstein's post-Tractarian philosophy. In particular, I shall demonstrate that his intensional interpretation of infinity — or, rather, his opposition to the extensional understanding of the concept — is strictly revisionist in nature, and so its cogency as an argument against transfinite mathematics is limited.

### §3

Can Cantor's set theory be interpreted and accepted in a form devoid of mathematical Platonism, while not being reduced to its truncated, Wittgensteinian version consisting of a collection of rather uninteresting calculi? I will attempt to answer this question in the affirmative.

<sup>17</sup> As has been noted by Rodych (Rodych, 2000: 303), among others, this belief has roots in the *Tractatus*, 6.211.

<sup>18</sup> Similarly, in RFM: II, §12 Wittgenstein puts forward the rhetorical question: “What can the concept of ‘non-denumerable’ be used for?”

The justification of my answer will be twofold. First, let us adopt the perspective I will call quietistic conventionalism. The perspective broadly resembles Wittgenstein's formalist middle period developed during his discussions with Moritz Schlick and Friedrich Waismann as he became acquainted with Gottlob Frege's critique of Thomae's older formalism. Such a perspective allows us to view transfinite set theory as the result of a series of definitions established by arbitrary decisions. This is how we have arrived at concepts such as the equinumerability of infinite sets, denumerability, non-denumerability, cardinality of  $\aleph$ , cardinality of the power set of the  $\aleph_0$ -infinite set, continuum, etc. Subsequently, in the same manner, we have been led to agree on various equations, such as:

$$\aleph_n = 2^{\aleph_{n-1}},$$

$$c = \aleph_1,$$

$$\aleph_0^n = \aleph_0,$$

$$c^{\aleph_0} = c,$$

etc.

All of this can be treated as the grammatical rules of a given calculus. We can also arbitrarily recognise Dedekind's constructions, proofs by Cantor and others, and so on, regardless of whether they fulfil the condition of constructibility, no matter how they are understood. It is noticeable that the later Wittgenstein does indeed toys with such a *laissez-faire* approach to the set theory (RFM: II, §35, quoted in the previous section). The fact that such a calculus allows for contradictions is not a grave problem, since the perspective we have assumed allows us to deal with the eventual emergence of a contradiction by appending some specific rules to the calculus (WVC: 119f.). One could even say that the history of set theory *de facto* conforms to this model of development.

The quietist conventionalist approach invites the following objection: since all rules are accepted arbitrarily, on what grounds do we adopt this and not another syntax? How is it that mathematicians generally have come to approve of the series of arbitrary decisions made by the founders of set theory?

Note that this is a question of natural history, something factual; it is therefore empirical and not philosophical in nature. We do not ask about reasons but causes. This calls for an empirical answer in the form of exposed non-arbitrary, pre-grammatical, and pre-normative conditions for grammar itself. This will constitute the second stage of the rationale of my answer to the possibility of non-Platonist transfinite set theory.

In their 2000 book *Where mathematics comes from*, Lakoff and Núñez put forward a psychological explanation of the formation of the concept of actual infinity. To this end, they utilise conceptual metaphorization theory and several other theories proposed within cognitive linguistics. The explanation they provide lacks sophistication and has attracted criticism, especially, though not exclusively, from mathematicians who are committed Platonists. It also lacks

originality: one could describe it as a modern articulation of beliefs present within various currents of finitism and intuitionism since at least the 1920s. Nonetheless, their explanation has maintained its position in the cognitive sciences for twenty years without any serious alternative.

The authors of *Where mathematics comes from* have advanced a hypothesis according to which actual infinity is a metaphorical concept in the strict, technical sense of the term. “Literally”, they write, “there is no such thing as the result of an endless process. If a process has no end, there can be no ‘ultimate result.’ But the mechanism of metaphor allows us to conceptualise the ‘result’ of an infinite process — in the only way we have for conceptualising the result of a process — that is, in terms of a process that does have an end” (Lakoff & Núñez, 2000: 158).<sup>19</sup> A metaphor in this context signifies a projection linking two conceptual domains, allowing us to enrich the conceptual structure of the target domain with features already present within the source domain (Evans, 2007: 136f.). Thus, when Lakoff and Núñez describe the Basic Metaphor of Infinity (henceforth, BMI), they refer to the result of mapping the domain of completed (finite) iterative processes onto the domain of infinite iterative processes. While some features of the result are shared by both domains (both processes must have a starting point, their initial iteration leads to a first state, and there is a general notion of movement from one state to another), the crucial feature is the product of the projection: there exists a unique and final resultant of an indefinite process, following every non-final state. According to the mentioned authors, this is how we arrive at the concept of infinity. They stipulate that all cases of that concept’s use in mathematics can be explained in terms of the BMI (Lakoff & Núñez, 2000: 159–161). Indeed, they devote individual chapters to explaining the role that the BMI plays in our conceptualisation of real numbers (Lakoff & Núñez, 2000: 181–207) and how it forms the basis of transfinite set theory (Lakoff & Núñez, 2000: 208–222). However, I do not wish to dedicate space here for an in depth discussion of their conception; I only mention that its details are highly disputable. But from a philosophical point of view these details are insignificant and may be even false: what is of interest to us is the very existence of an explanation in such a form, since it allows us to consider the problem of actual infinity from a new angle.

In general, Lakoff and Núñez’s conception — or any similar attempt for that matter — boils down to providing a strictly finitist, yet simultaneously non-disqualifying explanation of the emergence and operation of transfinite calculus. Such an account does not postulate the existence of infinite expansion or transfinite power sets, nor does it refer to any special mathematical

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<sup>19</sup> It is worth noting that the authors remark that the category of iterative process along with its subcategories of completed and indefinite processes is embodied in the sense that it emerges naturally as cognitive account of bodily activities like breathing or walking (Lakoff & Núñez, 2000: 155–157).

intuition: from the psychological perspective presented by the authors of *Where mathematics comes from*, Cantor's set theory is undoubtedly an invention and not a discovery. On the other hand, their account — and, generally, any account of this kind — cannot be described as revisionist, since it is very far from postulating prohibition of any axioms. It is also compatible with both ZF and ZFC systems, although the axiom of infinity, as well as all other axioms it treats upon, are not to be understood as postulating existence in the sense of ontological commitment, but rather as permission to use certain symbols when building mathematical constructions that have a carefully defined function within our calculus to signify magnitudes, we are unable to explicitly give.

One caveat must be given here: Lakoff and Núñez understand the term “concept” very differently from Wittgenstein. From the perspective of cognitive linguistics, “conceptual structure” denotes a certain hypothetical cognitive mechanism: a piece of equipment allowing us to carry out mental operations. Let us not be misled by similarities in expression: while Wittgenstein also metaphorically compares concepts to tools (PI: §11), elsewhere he refers to them as something normative and communal. His concepts are dependent on grammar, understood as a system of rules functioning on a level above the cognitive structures of a single mind. Lakoff and Núñez seem to describe only an aspect of what Wittgenstein calls a concept; he would have considered what in cognitive linguistics is now called conceptual structures to be factually grounded patterns of disposition to particular forms of categorisation. The examination of such structures does not lie in the domain of philosophical analysis, but in that of empirical sciences. Likewise, the BMI hypothesis is decidedly empirical in nature: it can be tested and falsified through empirical experiments. Consequently, to use Wittgenstein's metaphor, Lakoff and Núñez continue with their inquiry at a depth where the philosopher's spade is turned, for it has reached bedrock (PI: §217). The BMI itself is a causal explanation of the agreement in judgments among mathematicians engaged in set theory research (PI: §242).

In order to move away from devices such as the BMI to conceptual structures as understood by Wittgenstein, we would need to establish and implement some grammar among our peers. The establishment of a grammar is always a certain invention. Wittgenstein discusses this in a sense using the example of a fairy tale in which the princess, asked by the king to arrive neither naked nor dressed, gets the idea to wear fishnet (WLC1: 185f.). Initially, we are perplexed, since we lack the procedure to reach the required solution, but as we come up with an adequate process, it appears to be obviously correct and becomes universally accepted.<sup>20</sup> Dedekind and Cantor, as well as Zermelo

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<sup>20</sup> The invention of a new calculus is a constantly recurring topic in Wittgenstein's mathematical considerations (Floyd, 2000: 244–252).



and Fraenkl, had originated such grammatical inventions, though they were convinced they were discovering and describing mathematical facts. Taking a Wittgensteinian stance on conceptual structures (while setting aside his revisionary tendencies), those inventions are on one hand revealed as arbitrary, while on the other they manifest themselves as secured in universal, pre-grammatical cognitive conditions which enable mathematicians' agreement in judgements.

However, one cannot neglect to mention that in this case there can be no talk of unequivocal agreement in judgements, since there are many standpoints in direct opposition to set theory.<sup>21</sup> One could respond to this objection by pointing to the two main reasons for that discordance. First, to this point neither transfinite calculus nor actual infinity have proven beyond reasonable doubt to be indispensable to any extra-mathematical applications. Second, the underlying context of the invention of transfinities was mathematical Platonism; therefore, it seems that acceptance of the techniques proposed by Cantor *et consortes* automatically results in serious ontological commitments (this is a belief readily reinforced by the many contemporary set theorists who are dedicated Platonists). The first reason will remain valid for as long as there is no application of ZFC axioms in physics or technology.<sup>22</sup> As has been demonstrated above, the second reason is partly illusory: one does not need to assume that symbols and notions employed in calculi refer to anything real. The continuum may exist in precisely the same sense as a complex number with a non-zero imaginary component or the natural number 12. For Frege, the latter was a real object — the set of all 12-element sets existing in the “third realm” — but this is merely a variation on the Platonic theme that we do not need to follow. For Wittgenstein, on the other hand, the number 12 can be given thanks to a surveyable symbolic construction<sup>23</sup>. I believe the

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<sup>21</sup> Indeed, there have been many examples of disputes concerning the introduction of new concepts, techniques, or sets of symbols throughout the history of mathematics. Examples include the introduction of Arabic numerals into Europe, the emergence of zero, and the dispute between the abacists and the algorists. It seems to be that it was the effectivity and applicability of a given technique that dispelled controversy for good (Ifrah, 2000: 586–590).

<sup>22</sup> Some aspects of transfinite set theory do indeed seem completely useless. The invention of a ball-duplicating machine that would work on the basis of the Banach-Tarski theorem seems rather unlikely. It is also hard to imagine that the CH could ever find any empirical confirmation or falsification by appearing to be true or false in a manner similar to Euclid's fifth postulate. On the other hand, some scholars have argued that set theory does indeed have some applications in physics (Putnam, 2007).

<sup>23</sup> Even before starting to write the *Tractatus*, Wittgenstein was already extremely critical of the basic assumptions of Russellian and Fregean logicism. Not only did he reject their ontology and epistemology of mathematics, but he also dismissed the very notion of founding mathematics upon set theory. He considered the latter unable to provide an adequate grounding for the former, since any attempt at rendering numbers as individual objects obscures their

conclusion to be drawn from the existence of the BMI theory should be that actual infinity can also be surveyable, provided we accept the key element in the construction of the relevant metaphor. This is obviously a surveyability that the revisionist Wittgenstein would not accept, for it is predicated upon an approach that he explicitly condemned: the understanding of an unlimited process, defined by a certain law, as a thing, or, more specifically, a container (Lakoff & Núñez, 2000: 163).

#### §4

The key property of the BMI — that an indefinite process can have a final unique resultant state — is precisely what Ramsey and Wittgenstein considered to be absurd, instead opting for an intensional (*de facto* Aristotelian) conception of infinity. This revisionist, intensional philosophy of mathematics developed by Wittgenstein over his middle and late period remains a fascinating and attractive approach to set theory. Its revisionism rests upon two criteria: that mathematics is to be grounded in extra-mathematical application, while its admissible extensions must conform to strictly finitist surveyability. Both conditions posit certain philosophical declarations with far-reaching ramifications.

Can such conditions be justified? This question can likewise be considered on the two separate levels. Having adopted the quietist conventionalism outlined in §3, on a philosophical level we can interpret them as the grammar arbitrarily established by Wittgenstein for the notions of “infinity” and “mathematics”. Such a grammar would be internally coherent and in line with the general anti-metaphysical approach espoused by him. On the other hand, the question concerning the origin of the specific form of this grammar, along with the broader issue related to it, namely why Aristotle, Hume, Kronecker, Ramsey, Wittgenstein, and numerous others chose to reject the notion of actual infinity, may be considered from an empirical point of view. Perhaps there exist some “hard” biological preconditions for mistrust of actual infinity; then again they could be limited to ones based in culture and history.

Regardless of the reasons for favouring the intensional grammar of infinity by Wittgenstein (*et consortes*), at the rational level its arbitrariness means that its applicability as an argument against transfinitism is limited to the very fact that the latter finds a legitimate rival, implying that the set-theoretic approach to mathematics has an interesting and viable alternative. However, it also means that Wittgenstein’s position is not unlike that of Moore in his dispute with a king who believes that the world had not existed prior to his birth,

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defining trait, which is that they belong to a sequence, or, to phrase it in a Tractarian manner, they are the exponents of an abstract operation iterated without an end (TLP: 6.021).

an example described in *On certainty*. For the king to be convinced, he would have to “be brought to look at the world in a different way”, says Wittgenstein, suggesting that George E. Moore would eventually be in a position to do so, though he would not be able to utilise any of his intra-systemic arguments to support his view (OC: §92). What could the English philosopher use as his last resort? I suppose Wittgenstein's suggestion would have been an appeal to practice, the applicability of the system to people's lives.

Wittgenstein apparently believed that the same sorts of arguments could settle the matter of infinity in his favour: all conceivable extra-mathematical applications of the notion would fall under his intensional conception, with Cantor's transfinite hierarchy having no imaginable use. Let us leave aside the question of whether this is actually the case or not.<sup>24</sup> Instead, let us notice that the applicability of a calculus is in fact an accidental matter that seems to be external to mathematics itself. Moreover, it is plausible that though a certain collection of mathematical techniques will never find its application, it can well serve as crucial inspiration for the invention of other, easily applicable techniques.<sup>25</sup> Therefore, Wittgenstein's criterion of applicability is revealed as too strict to be reasonable. A lack of contemporary application, or even a clue of what a future application might be, should not then be taken as a decisive argument in support of or against a given set of mathematical techniques.

## CONCLUSION

And so we arrive at two alternative proposals for the understanding of infinity: no-metaphysical-strings-attached transfiniteism and intensional finitism. They differ not only by the techniques they permit, but first and foremost by the criteria according to which they acknowledge something as an element of mathematics. Both proposals are legitimate from the stance that I have described as quietist conventionalism, which I associate with the non-revisionist tendency within Wittgenstein's thought. The perspective taken by the revisionist Wittgenstein accepts only the latter proposal (intensional finitism), but the two main arguments in favour of such a revisionism are rather unconvincing. First, the acceptance or rejection of the surveyability of the symbol of actual infinity is in fact our own decision, meaning that the intensional character of infinity is simply an internal postulate of Wittgenstein's theory of

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<sup>24</sup> Certainly, one could easily find some use for transfinite set theory: it may be employed to entertain students or to impress someone we fancy, but this is not the kind of use that we would consider relevant.

<sup>25</sup> This, on the other hand, is a form of non-direct application that we would consider relevant (see the previous footnote).

mathematics. Second, the reliance upon applicability (or even the possibility of it) as a criterion determining whether something belongs to mathematics or is just a game of symbols is highly dubious, since it presupposes a static view of science: we are unable to foresee what will prove useful in future. Something currently considered to be mere fiction may become the source of inspiration for a new path of scientific research, just like atomism, which the majority of late nineteenth-century physicists considered to be metaphysical nonsense. Therefore, we arrive at one conclusion: that our attempts at weeding out metaphysics from formal science, however reasonable, should not constrain too strictly the pursuit for the proper sense of certain concepts, even though they may be of metaphysical inspiration.

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